stiffness matrix, and this shift would not allow the damage location to be determined. However, in the closed-loop plots the impact of damage is more statistically significant, showing that not only is the stiffness matrix clearly changed, but that it is most likely k_3 that is changed. The test statistic for K_{11} is small, indicating no significant difference in mean values, for both the open-loop and closed-loop cases, as expected, because damage in k_3 does not affect K_{11}. The test statistic for K_{22} and K_{33} is substantially larger for closed-loop SMA than for the open-loop case, indicating that the means are significantly different.

The output feedback results, shown in Fig. 2, are not quite as distinct as in the full-state feedback case, but it is still clear that the sensitivity-enhanced case outperforms the open-loop case. As before, damage is only discernable in the third diagonal element for the open-loop plots, whereas, in the closed-loop data, all three diagonal elements are significantly changed, which makes the presence of damage more easily determined, but still does not permit damage localization.

The output feedback approach is applicable to more complex systems; however, ability to determine the location of the damage will depend heavily on the number and location of sensors.

IV. Conclusions

SMA greatly enhances the shifts in the effective stiffness matrix caused by a small change in the actual stiffness of an individual spring. It has also been shown that the position output feedback case performs nearly as well as the full-state feedback case. SMA provides a means for designing sensitivity enhancing controllers for use in the identification of stiffness damage in simple structures.

Acknowledgments

This research is supported by the National Science Foundation under Grant CMS-9988414 and by NASA under NGT5-40088. The authors are grateful for the support.

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Thermal Postbuckling of Uniform Spring-Hinged Columns Using a Simple Method

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Nomenclature

A	=	area of cross section
a	=	lateral displacement at the middle of the column
c	=	as defined in Eq. (41)
E	=	Young's modulus
I	=	area moment of inertia
k_1, k_2	=	stiffnesses of the rotational spring (Fig. 1)
L	=	length of the column
P	=	axial compressive load
P_{cr}	=	critical load
P_{NL}	=	postbuckling load
r	=	radius of gyration
T	=	temperature
T_a	=	axial tension developed as a result of large deformations
T_{cr}	=	critical temperature
U	=	strain energy
u	=	axial displacement
W	=	work done by the load P
w	=	lateral displacement
x	=	axial coordinate
α	=	coefficient of linear thermal expansion
β_1, β_2	=	coefficients in Eq. (4)
γ_1, γ_2	=	rotational spring stiffness parameters, $(k_1L/EI, k_2L/EI)$
λ	=	load parameter, (PL^2/EI)
λ_{cr}	=	critical load parameter, $(P_{cr}L^2/EI)$
λ_{NL}	=	postbuckling load parameter, $(P_{NL}L^2/EI)$
λ_{Ta}	=	axial tension parameter, (T_aL^2/EI)

Superscript

$()'$	=	differentiation with respect to x

Introduction

IN a recent Note,[1] the authors have proposed a simple intuitive method to predict the thermal postbuckling behavior of uniform columns, with axially immovable ends, with classical boundary conditions like pinned or clamped at the ends. This method, basically, requires the linear thermal buckling load and the tension developed in the column caused by large deformations.

In actual practice, the classical boundary condition like pinned and clamped are difficult to realize, and the end conditions will be somewhere in between the pinned and clamped ones. This situation can be simply represented by applying elastic rotational springs with a specified stiffness and making the lateral displacements at the ends zero. The limiting cases of pinned and clamped conditions can be obtained by making the rotational spring stiffness either zero or very large.

Received 5 February 2003; revision received 1 August 2003; accepted for publication 1 September 2003.

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The postbuckling behavior of spring-hinged columns can be obtained using a powerful numerical method like the finite element method.[2] But, accurate analytical solutions for this problem are very much useful for studying the effects of various parameters involved with very little computational effort.

In this Note the thermal postbuckling behavior of spring-hinged (with end elastic restraints and zero lateral displacements) uniform columns is studied using the simple, intuitive method[1] proposed by the authors, using the method of Elishakoff[3] to assume the admissible lateral displacements.

Present Method

The simple, intuitive method[1] to predict the thermal postbuckling behavior of columns is briefly described here for the sake of completeness and it involves the following two steps:

1) If the column with axially immovable ends is subjected to a temperature rise T from the stress-free temperature, a mechanical compressive load $P(=AE\alpha T)$ develops in the column. The column buckles at $T_{\rm cr}$ when the mechanical load becomes the Euler buckling load $P_{\rm cr}$.

2) Any further increase to temperature from $T_{\rm cr}$, say, by ΔT, the column undergoes lateral deformation with a specific central deflection corresponding to ΔT because of the nonlinearity involved in the strain displacement relations. The ΔT can be represented by an equivalent mechanical load ΔP. An axial tension Ta is developed in the column because of the large deformations and immovable ends, and it is numerically equal to ΔP. This axial tension contributes to the postbuckling load $P_{\rm NL}$ and is equal to $P_{\rm cr} + \Delta P$, which can be expressed as $P_{\rm cr} + Ta$.

Based on the preceding two steps, we can write in the nondimensional form the postbuckling load parameter $\lambda_{\rm NL}$ as

$$\lambda_{\rm NL} = \lambda_{\rm cr} + \lambda_{\rm Ta} \tag{1}$$

or

$$\frac{\lambda_{\rm NL}}{\lambda_{\rm cr}} = 1 + \frac{\lambda_{\rm Ta}}{\lambda_{\rm cr}} \tag{2}$$

From Eq. (2), it is evident that the ratio of the postbuckling to the critical load (postbuckling behavior) can be obtained if one knows $\lambda_{\rm Ta}$ and $\lambda_{\rm cr}$. $\lambda_{\rm Ta}$ can be obtained, following Woinowsky–Krieger,[4] as

$$\lambda_{\rm Ta} = \frac{L}{2r^2}\int_0^L (w')^2\,{\rm d}x \tag{3}$$

Any trigonometric or algebraic function that satisfies the boundary conditions of the column can be used to compute $\lambda_{\rm Ta}$ in Eq. (3), and $\lambda_{\rm cr}$ can be evaluated by any analytical or numerical method or taken from literature.

Finally, it can be emphasized that in the present method to predict the thermal postbuckling behavior of columns it is sufficient if accurate values (expressions) for $\lambda_{\rm Ta}$ and $\lambda_{\rm cr}$ are available.

Application to the Spring-Hinged Uniform Columns

Figure 1 shows the uniform column with the lateral displacement w constrained and having elastic rotational springs of stiffness k_1 and k_2 at the ends. The column is subjected to a uniform temperature rise T from the stress-free state. As has been already mentioned, we need the values of the thermal linear buckling load and the tension developed in the column caused by large deformations ($\lambda_{\rm cr}$ and $\lambda_{\rm Ta}$ in the nondimensional form).

The $\lambda_{\rm cr}$ and $\lambda_{\rm Ta}$ are evaluated by assuming suitable algebraic functions for the lateral displacement w. In the present study a general two-term algebraic expression for w (which can be reduced to a one-term solution by deleting the second term) is assumed as

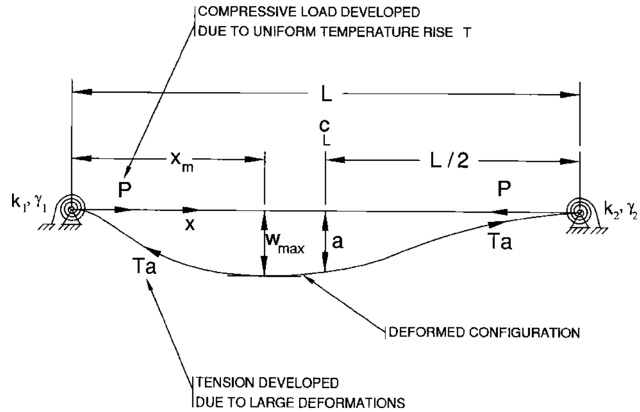

Fig. 1 Axially immovable uniform spring-hinged column.

$$w = \beta_1\left[\alpha_1\left(\frac{x}{L}\right) + \alpha_2\left(\frac{x}{L}\right)^2 + \alpha_3\left(\frac{x}{L}\right)^3 + \alpha_4\left(\frac{x}{L}\right)^4 + \alpha_5\left(\frac{x}{L}\right)^5\right] + \beta_2\left[\alpha_6\left(\frac{x}{L}\right) + \alpha_7\left(\frac{x}{L}\right)^2 + \alpha_8\left(\frac{x}{L}\right)^3 + \alpha_9\left(\frac{x}{L}\right)^5 + \alpha_{10}\left(\frac{x}{L}\right)^6\right] \tag{4}$$

Following Elishakoff[3] (to satisfy the spring-hinged boundary condition at $x = 0$ and L), the boundary conditions to be satisfied are

$$w(0) = 0, \qquad w(L) = 0 \tag{5}$$

$$EIw''(0) - k_1 w'(0) = 0, \qquad EIw''(L) + k_2 w'(L) = 0 \tag{6}$$

Noting that each term in the expression for w in Eq. (4) has to satisfy the boundary conditions given by Eqs. (5) and (6), we obtain

$$\alpha_1 = \frac{1}{D_1}\left[6 + \gamma_2 - (14 + 2\gamma_2)\frac{T_1}{T_2}\right] \tag{7}$$

$$\alpha_2 = \frac{\gamma\alpha_1}{2} \tag{8}$$

$$\alpha_3 = \frac{1}{D_1}\left[-(12 + 5\gamma_1 + 3\gamma_2 + \gamma_1\gamma_2) + \left(20 + 9\gamma_1 + 4\gamma_2 + \frac{3}{2}\gamma_1\gamma_2\right)\frac{T_1}{T_2}\right] \tag{9}$$

$$\alpha_4 = 1.0 \tag{10}$$

$$\alpha_5 = -\frac{T_1}{T_2} \tag{11}$$

$$\alpha_6 = \frac{1}{D_1}\left[24 + 3\gamma_2 - (14 + 2\gamma_2)\frac{T_3}{T_2}\right] \tag{12}$$

$$\alpha_7 = \frac{\gamma_1\alpha_6}{2} \tag{13}$$

$$\alpha_8 = \frac{1}{D_1}\left[\left(20 + 9\gamma_1 + 4\gamma_2 + \frac{3}{2}\gamma_1\gamma_2\right)\frac{T_3}{T_2} - (30 + 14\gamma_1 + 5\gamma_2 + 2\gamma_1\gamma_2)\right] \tag{14}$$

$$\alpha_9 = -\frac{T_3}{T_2} \tag{15}$$

$$\alpha_{10} = 1.0 \tag{16}$$

where

$$T_1 = (6+\gamma_2)(1+\gamma_1 x_m) - 3x_m^2(12+5\gamma_1+3\gamma_2+\gamma_1\gamma_2) + 4x_m^3 D_1 \tag{17}$$

$$T_2 = (14+2\gamma_2)(1+\gamma_1 x_m) - 3x_m^2\left(20+9\gamma_1+4\gamma_2+\frac{3}{2}\gamma_1\gamma_2\right) + 5x_m^4 D_1 \tag{18}$$

$$T_3 = (24+3\gamma_2)(1+\gamma_1 x_m) - 3x_m^2(30+14\gamma_1+5\gamma_2+2\gamma_1\gamma_2) + 6x_m^5 D_1 \tag{19}$$

with x_m being the point of maximum deflection or zero slope,

$$\gamma_1 = \frac{k_1 L}{EI}, \qquad \gamma_2 = \frac{k_2 L}{EI} \tag{20}$$

$$D_1 = 6 + 2\gamma_1 + 2\gamma_2 + \frac{\gamma_1\gamma_2}{2} \tag{21}$$

The linear critical thermal load parameter λ_{cr} is obtained by using the classical Rayleigh–Ritz method as

$$\frac{\partial}{\partial\beta_1}(U - W) = 0 \tag{22}$$

$$\frac{\partial}{\partial\beta_2}(U - W) = 0 \tag{23}$$

where the strain energy U and the work done W are given by

$$U = \frac{EI}{2}\int_0^L (w'')^2\,\mathrm{d}x + \frac{k_1}{2}w'(0)^2 + \frac{k_2}{2}w'(L)^2 \tag{24}$$

$$W = \frac{P}{2}\int_0^L (w')^2\,\mathrm{d}x \tag{25}$$

Solving the resulting linear simultaneous equations in β_1 and β_2 obtained from Eqs. (22) and (23), we get the characteristic equation for the load parameter λ as

$$\lambda^2\bar{c} - \lambda\bar{d} + \bar{e} = 0 \tag{26}$$

with

$$\bar{c} = A_4 A_5 - A_6^2 \tag{27}$$

$$\bar{d} = A_4\left(A_2 + \gamma_1\alpha_6^2 + \gamma_2 C_2^2\right) + A_5\left(A_1 + \gamma_1\alpha_1^2 + \gamma_2 C_1^2\right) - 2A_6(A_3 + \gamma_1\alpha_1\alpha_6 + \gamma_2 C_1 C_2) \tag{28}$$

$$\bar{e} = \left(A_1 + \gamma_1\alpha_1^2 + \gamma_2 C_1^2\right)\left(A_2 + \gamma_1\alpha_6^2 + \gamma_2 C_2^2\right) - (A_3 + \gamma_1\alpha_1\alpha_6 + \gamma_2 C_1 C_2)^2 \tag{29}$$

The expressions for $A_1 - A_6$, C_1, C_2 are

$$A_1 = \frac{144}{5} + \alpha_2(4\alpha_2 + 12\alpha_3 + 16 + 20\alpha_5) + \alpha_3(12\alpha_3 + 36 + 48\alpha_5) + \alpha_5\left(\frac{400}{7}\alpha_5 + 80\right) \tag{30}$$

$$A_2 = 100 + \alpha_7(4\alpha_7 + 12\alpha_8 + 20\alpha_9 + 24) + \alpha_8(12\alpha_8 + 48\alpha_9 + 60) + \alpha_9\left(\frac{400}{7}\alpha_9 + 150\right) \tag{31}$$

$$A_3 = \frac{360}{7} + 8\alpha_7 + 18\alpha_8 + 40\alpha_9 + \alpha_2(4\alpha_7 + 6\alpha_8 + 10\alpha_9 + 12) + \alpha_3(6\alpha_7 + 12\alpha_8 + 24\alpha_9 + 30) \tag{32}$$

$$A_4 = \frac{16}{7} + \alpha_1(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2 + 2\alpha_5) + \alpha_2\left(\frac{4}{3}\alpha_2 + 3\alpha_3 + \frac{16}{5} + \frac{10}{3}\alpha_5\right) + \alpha_3\left(\frac{9}{5}\alpha_3 + 4 + \frac{30}{7}\alpha_5\right) + \alpha_5\left(\frac{25}{9}\alpha_5 + 5\right) \tag{33}$$

$$A_5 = \frac{36}{11} + \alpha_6(\alpha_6 + 2\alpha_7 + 2\alpha_8 + 2\alpha_9 + 2) + \alpha_7\left(\frac{4}{3}\alpha_7 + 3\alpha_8 + \frac{10}{3}\alpha_9 + \frac{24}{7}\right) + \alpha_8\left(\frac{9}{5}\alpha_8 + \frac{30}{7}\alpha_9 + \frac{9}{2}\right) + \alpha_9\left(\frac{25}{9}\alpha_9 + 6\right) \tag{34}$$

$$A_6 = \frac{24}{9} + \alpha_1(\alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + 1) + \alpha_2\left(\alpha_6 + \frac{4}{3}\alpha_7 + \frac{3}{2}\alpha_8 + \frac{5}{3}\alpha_9 + \frac{12}{7}\right) + \alpha_3\left(\alpha_6 + \frac{3}{2}\alpha_7 + \frac{9}{5}\alpha_8 + \frac{15}{7}\alpha_9 + \frac{9}{4}\right) + \alpha_6 + \frac{8}{5}\alpha_7 + 2\alpha_8 + \frac{5}{2}\alpha_9 + \alpha_5\left(\alpha_6 + \frac{5}{3}\alpha_7 + \frac{15}{7}\alpha_8 + \frac{25}{9}\alpha_9 + 3\right) \tag{35}$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4 + 5\alpha_5 \tag{36}$$

$$C_2 = \alpha_6 + 2\alpha_7 + 3\alpha_8 + 5\alpha_9 + 6 \tag{37}$$

λ_{cr} can be obtained by solving Eq. (26) for the lowest value of λ as

$$\lambda_{cr} = \frac{\bar{d} - \sqrt{\bar{d}^2 - 4\bar{c}\bar{e}}}{2\bar{c}} \tag{38}$$

The axial tension parameter can be calculated by evaluating the eigenvector $\{\beta_1\beta_2\}^T$, so that $w = a$ at $x = L/2$ (as indicated in Fig. 1) and using the resulting displacement distribution for w in Eq. (3).

The axial tension parameter λ_{Ta} is obtained from Eq. (3) as

$$\lambda_{Ta} = \frac{1}{2}\left(\frac{a}{r}\right)^2\left[A_4 + f^2 A_5 + 2f A_6\right] \tag{39}$$

where

$$f = \frac{\lambda_{cr}A_4 - A_1 - \gamma_1\alpha_1^2 - \gamma_2 C_1^2}{A_3 + \gamma_1\alpha_1\alpha_6 + \gamma_2 C_1 C_2 - \lambda_{cr}A_6} \tag{40}$$

The expression for $\lambda_{NL}/\lambda_{cr}$ can be obtained from Eq. (2).

Numerical Results and Discussion

Using the formulation presented in the preceding section, the thermal postbuckling behavior of uniform spring-hinged columns has been evaluated in terms of λ_{cr} and c, where

$$\frac{\lambda_{NL}}{\lambda_{cr}} = 1 + c\left(\frac{a}{r}\right)^2 \tag{41}$$

However, before discussing the results of the spring-hinged columns the cases of classical boundary conditions like pinned–pinned columns (for which $\gamma_1 = \gamma_2 = 0$) and clamped–clamped columns (for which $\gamma_1, \gamma_2 \to \infty$) are considered. The postbuckling behavior of these columns using the Rayleigh–Ritz method is obtained using exact displacement distribution for w, given by Timoshenko and Gere[5] (trigonometric functions), and assuming suitable, compatible axial displacement u distributions, and results are presented

Table 1 Values of λ_{cr} and c of columns with different classical boundary conditions

	λ_{cr}			c		
Boundary conditions	Present method	Finite element method[a]	Rayleigh–Ritz method	Present method	Finite element method[b]	Rayleigh–Ritz method
Pinned–pinned ($\gamma_1 = \gamma_2 = 0$)	π^2[c]	9.8699	π^2[c]	0.25[c]	0.25	0.25[d]
Clamped–clamped ($\gamma_1 \to \infty$ and $\gamma_2 \to \infty$)	$4\pi^2$[e]	39.4985	$4\pi^2$[e]	0.0625[e]	0.0624	0.0625[f]
Pinned–clamped ($\gamma_1 = 0$ and $\gamma_2 \to \infty$)	20.1954[g]	20.2322	——	0.1472	0.1480	——

[a]Cubic displacement distribution for w (eight equal length elements in the column).
[b]Cubic displacement distributions for u and w (eight equal length elements in the column).
[c]$w = a \sin \pi x$.
[d]$w = a \sin \pi x$; $u = b \sin 2\pi x$.
[e]$w = a/2[1 - \cos 2\pi x]$.
[f]$w = a/2[1 - \cos 2\pi x]$; $u = b \sin 4\pi x$.
[g]Using the displacement distribution for w of the present study.

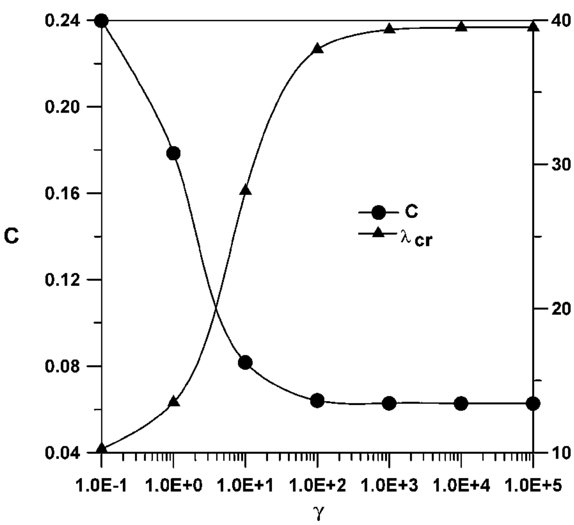

Fig. 2 Variation of C and λ_{cr} with γ ($\gamma_1 = \gamma_2 = \gamma$).

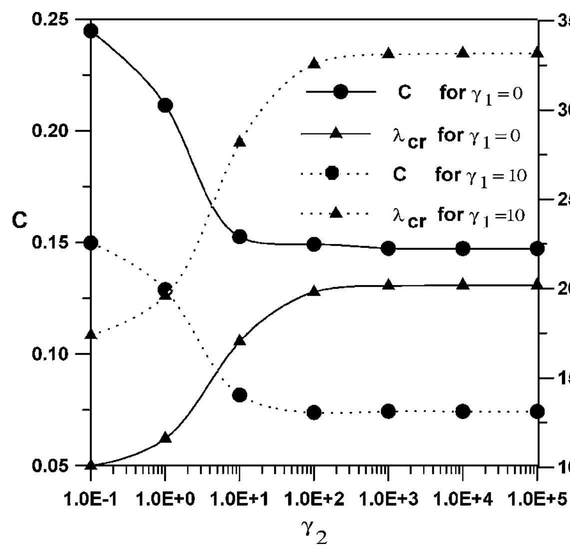

Fig. 3 Variation of C and λ_{cr} with γ_2 for a given γ_1.

in Table 1. Results through the finite element method with cubic displacement distributions for w and u over an element are also presented in Table 1. Results for the cases pinned–pinned and clamped–clamped columns, obtained by the present method with the w distributions of Timoshenko and Gere[5] (trigonometric functions), are also included in Table 1, and they show an excellent agreement with those of the Rayleigh–Ritz and the finite element methods. The present results for the case of pinned-clamped column (with $\gamma_1 = 0$ and $\gamma_2 \to \infty$) are also included in this table along with finite element results, and a very good agreement can be seen. In the present method the assumption of axial displacement mode is not necessary as in the case of the Rayleigh–Ritz and the finite element methods, which is a major advantage as it is not possible to assume a compatible axial displacement mode in the case of the spring-hinged columns. The assumed distributions for u are $\sin 2\pi x$ for the pinned–pinned columns and $\sin 4\pi x$ for the clamped–clamped columns. The difficulty can be understood in assuming u distribution for a spring-hinged columns with specific nonzero values of the rotational spring stiffness parameters γ_1 and γ_2. Figure 2 shows the plots of λ_{cr} and c of Eq. (41) for symmetric springs ($\gamma_1 = \gamma_2 = \gamma$) with various values of spring stiffness parameter γ. Figure 3 presents the plots of λ_{cr} and c for $\gamma_1 = 0.0$ and 10.0 and γ_2 varying between 10^{-1}–10^5. A close agreement of the values of c and λ_{cr} presented in Figs. 2 and 3 with those obtained by the finite element method is seen.

Conclusions

The thermal postbuckling behavior of uniform columns with end rotational elastic restraints and lateral displacement suppressed at the ends is studied in this Note using a simple method developed by the authors recently. The results obtained by the present simple method match excellently with the versatile finite element method. The primary advantage of the present simple method is that an assumption of the axial displacement, which is essential in the other methods like the Rayleigh–Ritz method and the finite element method, is not necessary, which makes the analysis very simple. Further, in the present method, to predict the thermal postbuckling behavior it is sufficient if one knows the linear thermal buckling load of the column and the tension developed in the column caused by large deformations.

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